



Ordinal potentials in smooth games

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Abstract

In the class of smooth non-cooperative games, exact potential games and weighted potential games are known to admit a convenient characterization in terms of cross-derivatives (Monderer and Shapley in *Games Econ Behav* 14:124–143, 1996a). However, no analogous characterization is known for ordinal potential games. The present paper derives necessary conditions for a smooth game to admit an ordinal potential. First, any ordinal potential game must exhibit pairwise strategic complements or substitutes at any interior equilibrium. Second, in games with more than two players, a condition is obtained on the (modified) Jacobian at any interior equilibrium. Taken together, these conditions are shown to correspond to a local analogue of the Monderer–Shapley condition for weighted potential games. We identify two classes of economic games for which our necessary conditions are also sufficient.

Keywords Ordinal potentials · Smooth games · Strategic complements and substitutes · Semipositive matrices

JEL Classification C6 · C72 · D43 · D72

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1 Introduction

In a potential game (Rosenthal 1973; Monderer and Shapley 1996a), players' preferences may be summarized in a single common objective function.¹ Knowing whether a specific game admits a potential can be quite valuable. For example, the existence of a potential reduces the problem of finding a Nash equilibrium to a straightforward optimization problem. Therefore, it is of some interest to know the conditions under which a potential exists. Clearly, sufficient conditions are most desirable. However, necessary conditions are also important. After all, such conditions may help avoiding a futile search for a potential. Moreover, as will be shown, necessary conditions may be indicative of sufficient conditions as well.²

This paper considers *smooth* games, i.e., non-cooperative n -player games with the property that strategy spaces are non-degenerate compact intervals and payoff functions are twice continuously differentiable (Vives 1999). In the class of smooth games, we derive simple necessary conditions for the existence of a generalized ordinal potential.³ Certainly, a generalized ordinal potential cannot exist if it is possible to construct a *strict improvement cycle* (Voorneveld 1997), i.e., a finite circular sequence of strategy profiles with the property that moving to the next profile in the sequence amounts to one player strictly raising her payoff by a unilateral change in strategy. However, in general, identifying strict improvement cycles in smooth games is not straightforward. We therefore consider a specific path in the neighborhood of a fixed strategy profile x_N^* , and shrink it to virtually infinitesimal size, as if using a pantograph.⁴ This differential approach leads to simple necessary conditions for the local feasibility of a generalized ordinal potential. In particular, it constitutes the basis of our main result, a local analogue of the Monderer–Shapley condition for the existence of an exact (or weighted) potential in smooth games.⁵

¹ Both exact and ordinal concepts have been considered in the literature. For a function P on the space of strategy profiles to be an *exact* potential (a *weighted* potential), the difference in a player's payoff resulting from a unilateral change of her strategy must precisely (up to a positive factor) equal the corresponding difference in P . For a function to be an *ordinal potential* (a *generalized ordinal potential*), any weak or strict gain (any strict gain) in a player's payoff resulting from a unilateral change of her strategy must be reflected by a corresponding gain in P .

² Applications of potential methods are vast and include, for example, the analysis of oligopolistic markets (Slade 1994), learning processes (Monderer and Shapley 1996b; Fudenberg and Levine 1998; Young 2004), population dynamics (Sandholm 2001, 2009; Cheung 2014), the robustness of equilibria (Frankel et al. 2003; Morris and Ui 2005; Okada and Tercieux 2012), the decomposition of games (Candogan et al. 2011), imitation strategies (Duersch et al. 2012), dynamics (Candogan et al. 2013a, b), equilibrium existence (Voorneveld 1997; Kukushkin 1994, 2011), solution concepts (Peleg et al. 1996; Tercieux and Voorneveld 2010), games with monotone best-response selections (Huang 2002; Dubey et al. 2006; Jensen 2010), supermodular and zero-sum games (Brânzei et al. 2003), and even mechanism design (Jehiel et al. 2008).

³ Since any ordinal potential game is, in particular, a generalized ordinal potential game, we also obtain necessary conditions for the existence of an ordinal potential.

⁴ A *pantograph* is a mechanical drawing instrument that allows creating copies of a plan on a different scale.

⁵ To caution the reader, we stress that, despite a similarity in terminology, the present analysis is *not* directly related to the use of local potentials in the analysis of informational robustness (Morris 1999; Frankel et al. 2003; Morris and Ui 2005; Okada and Tercieux 2012).

The analysis starts by considering strict improvement cycles that involve two players only. In this case, the existence of a generalized ordinal potential is shown to imply that the product of the slopes of any two players' mutual local best-response functions (or, more generally, the product of the corresponding cross-derivatives) must be nonnegative at any strategy profile at which at least two first-order conditions hold in the interior. Thus, borrowing the terminology familiar from contributions such as Bulow et al. (1985), Amir (1996), Dubey et al. (2006), and Monaco and Sabarwal (2016), we obtain as our first main necessary condition that the game must exhibit pairwise strategic substitutes or complements at any such point.

Next, we consider strict improvement cycles that involve more than two players. In the simplest case, the path runs along the edges of a small rectangular box that contains a fixed strategy profile x_N^* at its center. To formalize the resulting condition on cross-derivatives, we introduce a modified Jacobian at x_N^* by replacing all diagonal entries of the Jacobian of the game by zero and by multiplying all entries above the diagonal with negative one. It turns out that the existence of a particular strict improvement cycle corresponds precisely to the property that the modified Jacobian is *semipositive* (Fiedler and Pták 1966).⁶ We transform these conditions in several steps, using a powerful recursive characterization of semipositivity due to Johnson et al. (1994), as well as the technique of flipping around individual strategy spaces (Vives 1990; Amir 1996). These steps lead us ultimately to the local analogue of the Monderer–Shapley condition.

Finally, we apply our necessary conditions to two selected classes of games that traditionally have been studied by economists. These are the lottery contest with heterogeneous valuations, and a differentiated Bertrand game with linear demand and quadratic costs. For these classes of games, our necessary conditions turn out to be also sufficient for the existence of a generalized ordinal potential.

Related literature. Complete characterizations of exact and generalized ordinal potential games are known for the class of finite games (Monderer and Shapley 1996a; Milchtaich 1996). For the class of strategic games, Monderer and Shapley (1996a) provided a complete characterization of exact potential games. Voorneveld and Norde (1997) have shown that a strategic game admits an ordinal potential if and only if there are no weak improvement cycles and an order condition is satisfied.⁷ Exact potential games admit a convenient characterization in the class of smooth games, i.e., in the class of games with interval strategy spaces and twice continuously differentiable payoff functions. Specifically, a smooth game admits an exact potential if and only if the Jacobian of that game, i.e., the matrix of cross-derivatives of players' payoff functions, is globally symmetric (Monderer and Shapley 1996a). This characterization extends in a straightforward way to weighted potential games. However, no *differentiable* characterization has been available up to this point for the classes of ordinal potential or generalized ordinal potential games.⁸

⁶ The relevant elements of the theory of semipositive matrices will be reviewed below.

⁷ For a rigorous statement of this important result, we refer the reader to Voorneveld and Norde (1997).

⁸ Monderer and Shapley (1996a, p. 135) wrote: “Unlike (weighted) potential games, ordinal potential games are not easily characterized. We do not know of any useful characterization, analogous to the one given in (4.1), for differentiable ordinal potential games.” Since then, the problem has apparently remained unaddressed. See, e.g., the recent surveys by Mallozzi (2013), González-Sánchez and Hernández-Lerma (2016), or Lã et al. (2016).

The rest of the paper is structured as follows. Section 2 contains preliminaries. Section 3 discusses strict improvement cycles involving two players. The case of more than two involved players is dealt with in Sect. 4. Section 5 presents the local analogue of the Monderer–Shapley condition. Section 6 discusses boundary equilibria and related issues. Applications are provided in Sect. 7. Section 8 concludes. All the proofs have been relegated to an Appendix.

2 Preliminaries

Let Γ be a (*non-cooperative*) n -player game with set of players $N = \{1, \dots, n\}$, strategy space X_i for each player $i \in N$, and payoff function $u_i : X_N = X_1 \times \dots \times X_n \rightarrow \mathbb{R}$ for each $i \in N$. Given a strategy profile $x_N \equiv (x_1, \dots, x_n) \in X_N$, we denote by $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ the profile composed of the strategies chosen by the opponents of player i , so that $x_{-i} \in X_{-i} = X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_n$. Thus, $u_i(x_N) = u_i(x_i, x_{-i})$, etc. The game Γ will be called *smooth* (Vives 1999) if for any $i \in N$, the set X_i is a non-degenerate compact interval $X_i = [\underline{x}_i, \bar{x}_i] \subseteq \mathbb{R}$, with $\underline{x}_i < \bar{x}_i$, and u_i is twice continuously differentiable on X_N . Further, to denote the *interior* of the interval X_i , we will use the notation $\mathring{X}_i = (\underline{x}_i, \bar{x}_i)$.⁹

Following Monderer and Shapley (1996a), a non-cooperative game Γ is called an *ordinal potential game* [a *generalized ordinal potential game*] if there exists a function $P : X_N \rightarrow \mathbb{R}$ such that

$$u_i(x_i, x_{-i}) > u_i(\hat{x}_i, x_{-i}) \quad \text{if and only if} \quad P(x_i, x_{-i}) > P(\hat{x}_i, x_{-i}), \quad (1)$$

[implies]

for any $i \in N$, $x_i \in X_i$, $\hat{x}_i \in X_i$, and $x_{-i} \in X_{-i}$. Note that no continuity nor differentiability assumptions are imposed on the (generalized) ordinal potential functions.¹⁰

Given an n -player game Γ , a *strict improvement cycle* of length L is a finite non-degenerate sequence of strategy profiles

$$\dots \rightarrow x_N^0 \rightarrow x_N^1 \rightarrow \dots \rightarrow x_N^{L-1} \rightarrow \dots \quad (2)$$

in X_N with the property that, for any $l = 0, \dots, L-1$, there is a player $i = i(l) \in N$ such that $x_{-i}^{l+1} = x_{-i}^l$ and $u_i(x_i^{l+1}, x_{-i}^l) > u_i(x_i^l, x_{-i}^l)$, where x_N^L should be read as x_N^0 .

As pointed out by Voorneveld and Norde (1997), the existence of an ordinal potential in a given game is, under some additional condition, equivalent to the non-existence of a *weak* improvement cycle (wherein payoff has to increase strictly for

⁹ Thus, we restrict attention to one-dimensional strategy spaces. The extension to multi-dimensional strategy spaces is discussed in the conclusion.

¹⁰ This point is worth being mentioned because, as pointed out by Voorneveld (1997), a continuous ordinal potential game need not, in general, admit a continuous ordinal potential. See also Peleg et al. (1996).

some path transition only, and weakly otherwise). In the present analysis, we will need only the following necessary condition.

Lemma 1 (Voorneveld 1997) *Let Γ be a generalized ordinal potential game. Then, there is no strict improvement cycle in Γ .*

3 Strict improvement cycles involving two players

We start the analysis by considering local strict improvement cycles that involve precisely two players. For this case, the approach outlined in the introduction leads to the following observation.

Proposition 1 *Suppose that the smooth n -player game Γ admits a generalized ordinal potential. Then, for any two players $i, j \in N$ with $i \neq j$, it holds that*

$$\left\{ x_i \in \mathring{X}_i, x_j \in \mathring{X}_j, \quad \text{and} \quad \frac{\partial u_i(x_N)}{\partial x_i} = \frac{\partial u_j(x_N)}{\partial x_j} = 0 \right\} \Rightarrow \frac{\partial^2 u_i(x_N)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N)}{\partial x_i \partial x_j} \geq 0. \quad (3)$$

Thus, if a smooth n -player game admits a generalized ordinal potential, then at any strategy profile at which marginal payoffs vanish in the interior for two players, the product of the corresponding cross-derivatives is nonnegative. In particular, any smooth generalized ordinal potential game necessarily exhibits pairwise strategic substitutes or complements at any interior Cournot–Nash equilibrium.¹¹

Proposition 1 implies that games in which one player's best-response function is strictly increasing and another player's best-response function is strictly decreasing is never a generalized ordinal potential game. An example is the mixed oligopoly model by Singh and Vives (1984), in which one firm chooses price, and the other firm chooses quantity. Another instance is quantity competition between a dominant firm and several fringe firms (Bulow et al. 1985). Likewise, asymmetric contests (Dixit 1987) never admit a generalized ordinal potential. Many similar examples, taken from diverse areas such as law enforcement, business strategy, and citizen protests, can be found in Tombak (2006), Monaco and Sabarwal (2016), and Barthel and Hoffmann (2019), for instance.

In games with more than two players, condition (3) requires *pairwise* strategic complements or substitutes only. Therefore, unless sum-aggregative,¹² an ordinal potential game need not exhibit either strategic complements or strategic substitutes at x_N . Indeed, flipping around the natural order of the strategy space of one of three players, say, may certainly destroy the property of strategic complements or strategic

¹¹ Similar local necessary conditions may be obtained from the consideration of boundary equilibria, as will be explained in Sect. 6.

¹² See, e.g., Corchón (1994).

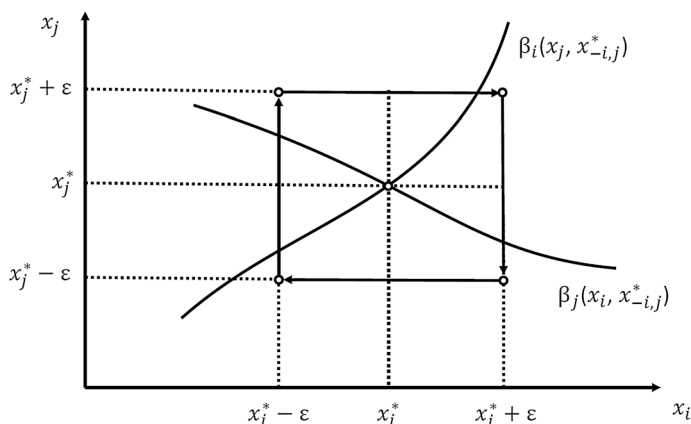


Fig. 1 Constructing a strict improvement cycle involving two players

substitutes, but it does not affect the pairwise property, nor does it affect the property of being an ordinal potential game.

To understand why Proposition 1 holds true, consider Fig. 1. Keeping the strategy profile $x_{-ij}^* = (x_1^*, \dots, x_{i-1}^*, x_{i+1}^*, \dots, x_{j-1}^*, x_{j+1}^*, \dots, x_n^*)$ fixed, player i 's best-response function¹³ $\beta_i = \beta_i(x_j, x_{-i,j}^*)$ is assumed to be well-defined and strictly *increasing* in player j 's strategy x_j , while player j 's best-response function $\beta_j = \beta_j(x_i, x_{-i,j}^*)$ is strictly *decreasing* in player i 's strategy x_i . Therefore,

$$\frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} < 0, \quad (5)$$

in conflict with the necessary condition. And indeed, for $\varepsilon > 0$ small enough, the circular path starting at the upper left corner of the depicted square and running clockwise around it, i.e., the finite sequence

$$\begin{aligned} \dots &\rightarrow (x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-ij}^*) \rightarrow (x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-ij}^*) \\ &\rightarrow (x_i^* + \varepsilon, x_j^* - \varepsilon, x_{-ij}^*) \rightarrow (x_i^* - \varepsilon, x_j^* - \varepsilon, x_{-ij}^*) \rightarrow \dots \end{aligned} \quad (6)$$

constitutes a strict improvement cycle, as will be explained now. To start with, consider the strategy change corresponding to the upper side of the square. Then, with ε

¹³ As usual, provided that $\partial^2 u_i / \partial x_i^2 < 0$ holds globally, player i 's marginal condition defines a function $\beta_i = \beta_i(x_{-i})$ that maps any vector $x_{-i} \in X_{-i}$ to a unique $\beta_i(x_{-i}) \in X_i$ such that $\partial u_i(\beta_i(x_{-i}), x_{-i}) / \partial x_i = 0$ holds in the interior. We will refer to β_i as player i 's *best-response function*. In particular, for any other player $j \neq i$, we may refer to

$$\sigma_{ij}(x_N) = \frac{\partial \beta_i(x_{-i})}{\partial x_j} = - \frac{\partial^2 u_i(x_N) / \partial x_j \partial x_i}{\partial^2 u_i(x_N) / \partial x_i^2} \quad (4)$$

as the *slope* of player i 's best-response function with respect to player j (in the interior). Thus, $\sigma_{ij}(x_N) \geq 0$ (≤ 0) if and only if Γ exhibits strategic complements (strategic substitutes) between i and j at x_N .

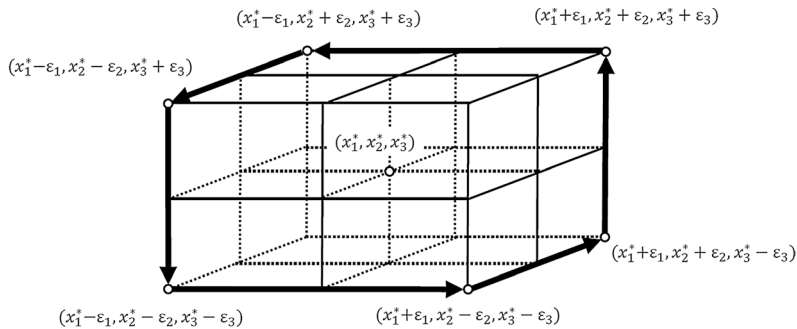


Fig. 2 Constructing a strict improvement cycle involving three players

small, player i 's payoff is first increasing (over a longer section of the side) and then decreasing (over a shorter section of the side).

The point to note is that, as a consequence of smoothness of payoffs at the Cournot–Nash equilibrium, player i 's payoff function along the upper side of the square may be approximated arbitrarily well by a parabola opening downwards, provided the square is small enough. As the parabola is symmetric around its peak, the payoff difference for player i , when switching from strategy $x_i - \epsilon$ to $x_i + \epsilon$, will be overall positive.¹⁴ Similar considerations apply to the remaining three sides of the square. In fact, at the bottom side, there is no trade-off because player i 's marginal payoff is always negative there. Thus, in sum, one may construct a strict improvement cycle that leads around the equilibrium. As noted in the previous section, however, this is incompatible with the existence of a generalized ordinal potential.

4 Strict improvement cycles involving more than two players

In this section, we consider strict improvement cycles that involve more than two players. We will initially restrict attention to a narrow class of paths in which the players involved consecutively raise their respective strategies, taking turns in their natural order, and subsequently lower their strategies, following the same order. Figure 2 illustrates a path of this kind for the case of three players.¹⁵

It turns out that the local feasibility of a strict improvement cycle is, for general $n \geq 2$, determined by the *semipositivity* (Fiedler and Pták 1966; Johnson et al. 1994) of a modified matrix that is derived from the Jacobian of the game by changing the sign of all entries above the diagonal, and by replacing all entries on the diagonal by zero.

¹⁴ There is a minor technical subtlety here in so far that the payoff difference approaches zero as ϵ goes to zero. However, as shown in the Appendix with the help of a careful limit consideration, the payoff difference approaches zero *from above* since the corresponding cross-derivative is positive. This turns out to be sufficient to settle the trade-off for a sufficiently small but still positive ϵ .

¹⁵ In contrast to the previous section, we will allow for edges that are not necessarily of equal length.

Formally, let $\lambda_N = (\lambda_1, \dots, \lambda_n)^T \in \mathbb{R}^n$ be a vector, where the T indicates transposition, as usual. We will write $\lambda_N > 0$ if all entries of λ_N are positive, i.e., if $\lambda_i > 0$ for all $i = 1, \dots, n$.

Definition 1 (Fiedler and Pták 1966) A square matrix $A \in \mathbb{R}^{n \times n}$ is *semipositive* if there exists $\lambda_N > 0$ such that $A\lambda_N > 0$.

Semipositivity generalizes the concept of a P-matrix (Gale and Nikaidô 1965). Intuitively, the fact that A is semipositive says that the interior of the convex cone generated by the column vectors of A intersects the positive orthant $\mathbb{R}_{++}^n = \{z_N \in \mathbb{R}^n : z_N > 0\}$. Thus, checking semipositivity of a matrix corresponds to solving a feasibility problem in linear programming. Moreover, semipositivity is a robust property, and one may think of it in the present context as a generalized sign test on matrix entries.

An analysis of the conditions for the described path to constitute a strict improvement cycle leads to the following observation.

Proposition 2 Suppose that the smooth n -player game Γ admits a generalized ordinal potential. Then, at any strategy profile $x_N \in X_N$ such that

$$x_1 \in \mathring{X}_1, \dots, x_n \in \mathring{X}_n, \quad \text{and} \quad \frac{\partial u_1(x_N)}{\partial x_1} = \dots = \frac{\partial u_n(x_N)}{\partial x_n} = 0, \quad (7)$$

the modified Jacobian

$$J(x_N) = \begin{pmatrix} 0 & -\frac{\partial^2 u_1(x_N)}{\partial x_2 \partial x_1} & \dots & -\frac{\partial^2 u_1(x_N)}{\partial x_n \partial x_1} \\ \frac{\partial^2 u_2(x_N)}{\partial x_1 \partial x_2} & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\frac{\partial^2 u_{n-1}(x_N)}{\partial x_n \partial x_{n-1}} \\ \frac{\partial^2 u_n(x_N)}{\partial x_1 \partial x_n} & \dots & \frac{\partial^2 u_n(x_N)}{\partial x_{n-1} \partial x_n} & 0 \end{pmatrix} \quad (8)$$

cannot be semipositive.

This result may be summarized as follows. Consider a smooth generalized ordinal potential game and take any interior strategy profile x_N at which marginal payoffs vanish for all players. For instance, x_N could be an interior Cournot–Nash equilibrium. Then, the modified Jacobian at x_N must not be semipositive.

There are two basic ways to generate even more stringent necessary conditions. First, one may permute the order in which players change their respective strategies. Second, one may flip around the natural order of individual strategy spaces. The general treatment of these extensions, accomplished in the working paper version (Ewerhart 2017, Sect. 4.2), turns out to be obsolete for the purpose of the present

analysis. However, we will use instances of permutations and flipped strategy spaces in the proof of our main result.¹⁶

5 A local analogue of the Monderer–Shapley condition

From the necessary conditions identified in the previous sections, one may derive the following theorem, which is the main result of the present paper.

Theorem 1 *Consider an interior strategy profile x_N^* in a smooth n -player game Γ such that*

$$\frac{\partial u_1(x_N^*)}{\partial x_1} = \dots = \frac{\partial u_n(x_N^*)}{\partial x_n} = 0, \quad (9)$$

and such that all cross-derivatives $\{\partial^2 u_i(x_N^)/\partial x_j \partial x_i\}_{i,j \in N \text{ s.t. } i \neq j}$ are nonzero. If Γ admits a generalized ordinal potential, then there exist positive weights $w_1(x_N^*) > 0, \dots, w_n(x_N^*) > 0$ such that*

$$w_i(x_N^*) \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} = w_j(x_N^*) \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} \quad (i, j \in N, j \neq i). \quad (10)$$

Thus, as outlined in the Introduction, the existence of an ordinal potential implies a local property that is analogous to the global differentiable condition for a weighted potential game.

To see the theorem at work, assume that payoffs admit the representation

$$u_i(x_N) = x_i \cdot \phi(x_N) \quad (i \in N), \quad (11)$$

where $X_i \subseteq \mathbb{R}_{++}$ for all $i \in N$, and $\phi : X_N \rightarrow \mathbb{R}$ is an arbitrary twice continuously differentiable function that does not depend on i . Then, as pointed out by Kukushkin (1994), the mapping

$$P(x_N) = x_1 \cdot \dots \cdot x_n \cdot \phi(x_N) \quad (12)$$

is an ordinal potential.¹⁷ Suppose that x_N^* is an interior Cournot–Nash equilibrium. Then, Theorem 1 implies the existence of weights $w_1(x_N^*) > 0, \dots, w_n(x_N^*) > 0$ such that (10) holds. This is indeed the case for

$$w_i(x_N^*) = \frac{1}{x_i^*}, \quad (13)$$

¹⁶ As will be explained in Sect. 6, no additional conditions can be obtained by considering strategy profiles at which less than two first-order conditions hold. Similarly, allowing for more complicated local paths does not tighten our conditions. For a discussion of this point, see likewise the working paper version (Ewerhart 2017).

¹⁷ For a recent application, see Nocke and Schutz (2018).

as can be checked in a straightforward way.¹⁸

Theorem 1 assumes that all cross-derivatives are non-vanishing. As the following example shows, this assumption cannot be dropped without losing the conclusion of Theorem 1.

Example 1 Let $N = \{1, 2\}$, $X_1 = X_2 = [-1, 1]$, and

$$u_1(x_1, x_2) = -(x_1 + x_2)^2, \quad (16)$$

$$u_2(x_1, x_2) = -(x_1 + x_2)^6. \quad (17)$$

This game is easily seen to admit the ordinal potential $P(x_1, x_2) = u_1(x_1, x_2)$. However, at the interior Cournot–Nash equilibrium $x_N^* = (0, 0)$, the cross-derivatives are given by

$$\frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} = -2 \quad \text{and} \quad \frac{\partial^2 u_2(x_N^*)}{\partial x_1 \partial x_2} = 0, \quad (18)$$

in conflict with relationship (10).

Thus, intuitively, the assumption that cross-derivatives do not vanish is needed because strictly monotone transformations of a player's payoff, under which the class of ordinal potential games is invariant, may possess points with zero slope.

The proof of Theorem 1 has three main steps. First, one considers a variety of strict improvement cycles involving either two or three players to show that, at any profile x_N^* , and for any set $\{i, j, k\} \subseteq N$ of pairwise different players,

$$\left\{ x_i^* \in \mathring{X}_i, x_j^* \in \mathring{X}_j, x_k^* \in \mathring{X}_k, \quad \text{and} \quad \frac{\partial u_i(x_N^*)}{\partial x_i} = \frac{\partial u_j(x_N^*)}{\partial x_j} = \frac{\partial u_k(x_N^*)}{\partial x_k} = 0 \right\} \\ \Rightarrow \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_k \partial x_j} \cdot \frac{\partial^2 u_k(x_N^*)}{\partial x_i \partial x_k} = \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} \cdot \frac{\partial^2 u_k(x_N^*)}{\partial x_j \partial x_k} \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_k \partial x_i}. \quad (19)$$

Then, an induction argument is used to obtain an analogous result for more than three players. In a final step, the weight for player i is defined as

¹⁸ Making use of player j 's first-order condition

$$x_j^* \frac{\partial \phi(x_N^*)}{\partial x_j} + \phi(x_N^*) = 0, \quad (14)$$

we see that

$$w_i(x_N^*) \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} = \frac{1}{x_i^*} \cdot \left\{ x_i^* \frac{\partial^2 \phi(x_N^*)}{\partial x_j \partial x_i} + \frac{\partial \phi(x_N^*)}{\partial x_j} \right\} = \frac{\partial^2 \phi(x_N^*)}{\partial x_j \partial x_i} - \frac{\phi(x_N^*)}{x_j^* x_i^*} \quad (15)$$

is symmetric with respect i and j . Thus, (10) indeed holds for the suggested local weights.

$$w_i(x_N^*) = \left(\left| \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} \right| \cdot \dots \cdot \left| \frac{\partial^2 u_{i-1}(x_N^*)}{\partial x_i \partial x_{i-1}} \right| \right) \cdot \left(\left| \frac{\partial^2 u_{i+1}(x_N^*)}{\partial x_i \partial x_{i+1}} \right| \cdot \dots \cdot \left| \frac{\partial^2 u_n(x_N^*)}{\partial x_{n-1} \partial x_n} \right| \right), \quad (20)$$

and the consistency relations derived in the previous two steps are used to show that these weights have the desired property (10).

6 Boundary equilibria and related issues¹⁹

While stated for interior strategy profiles at which all players have zero marginal incentives, it should be clear that Theorem 1 has immediate implications also for many strategy profiles x_N^* that do not satisfy these conditions. Indeed, it suffices to ignore any player whose strategy choice either exhibits strict marginal incentives or lies at the boundary (or both), and to consider the resulting game with less than n players. This trick, however, does not work for every strategy profile. Specifically, Theorem 1 does not deliver a local necessary condition at strategy profiles where either (1) at most one player's marginal incentives are zero, or (2) even though there are two or more players whose marginal incentives are zero, at most one of those chooses an interior strategy. In this section, we will discuss these cases.

We start by looking at the first case outlined above, i.e., we consider a strategy profile x_N^* at which at most one player's marginal incentives are zero. It turns out that, under such condition, there cannot be a strict improvement cycle locally at x_N^* . To the contrary, as the following result shows, it is then always feasible to locally construct an ordinal potential.

Proposition 3 *Let $x_N^* \in X_N$ be a strategy profile (possibly but not necessarily located at the boundary) in a smooth n -player game Γ , and let $i \in N$ be a player such that²⁰*

$$\frac{\partial u_j(x_N^*)}{\partial x_j} \neq 0 \quad (j \in N \setminus \{i\}). \quad (21)$$

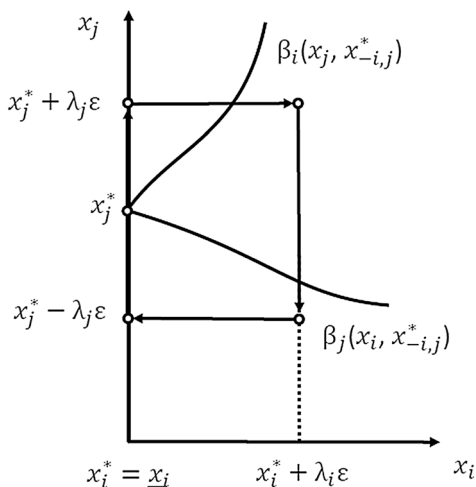
Then, Γ admits an ordinal potential in a small neighborhood of x_N^ .*

Thus, quite generally, strategy profiles that do not entail zero marginal incentives for at least two players cannot be used to identify local necessary conditions for the existence of an ordinal or generalized ordinal potential. In particular, this is so for equilibria in which at most one player chooses an interior strategy while all others choose a boundary strategy under positive shadow costs.

¹⁹ I am grateful for an anonymous reviewer for the suggestion to consider boundary equilibria in more detail.

²⁰ As usual, the derivative in (21) is understood to be one-sided if $x_j^* = \underline{x}_j$ or $x_j^* = \bar{x}_j$.

Fig. 3 Conditions at the boundary



The proof of Proposition 1 is based on a simple idea. Condition (21) implies that the payoff functions of all players except possibly player i are locally strictly monotone at x_N^* . Therefore, a natural candidate for an ordinal potential is a function $P(x_i, x_{-i})$ that adds together player i 's payoff function $u_i(x_i, x_{-i})$ and a suitable linear function in the vector x_{-i} . Indeed, it is not hard to see that, provided that the slope of the linear function with respect to x_j , for any $j \neq i$, is chosen sufficiently large in absolute terms so as to dominate the corresponding slope of $u_i(x_i, x_{-i})$ with respect to x_j , the function $P(x_i, x_{-i})$ becomes an ordinal potential for Γ in a small neighborhood of x_N^* .

The situation differs in the second case outlined above, i.e., when there are at least two players that have zero marginal incentives, but at most one of them chooses an interior strategy. Note that, in this case, a boundary strategy is chosen even though shadow costs vanish. Given the non-robustness of this type of situation, we confine ourselves to the analysis of an illustrative example involving two players, where one player chooses a (lower) boundary strategy and another player chooses an interior strategy.²¹ The consideration of a suitable family of strict improvement cycles of the shape outlined in Fig. 3 delivers the following result.

Proposition 4 *Suppose that the smooth n -player game Γ admits a generalized ordinal potential. Then, for any two players $i, j \in N$ with $i \neq j$,*

$$\left\{ x_i = x_i^*, x_j \in \dot{X}_j, \text{ and } \frac{\partial u_i(x_N)}{\partial x_i} = \frac{\partial u_j(x_N)}{\partial x_j} = 0 \right\} \\ \Rightarrow \left\{ \frac{\partial^2 u_i(x_N)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N)}{\partial x_i \partial x_j} \geq 0 \text{ or } \frac{\partial^2 u_j(x_N)}{\partial x_i \partial x_j} \cdot \frac{\partial^3 u_j(x_N)}{\partial x_j^3} \geq 0 \right\}. \quad (22)$$

²¹ Further generalizations are left for future work.

Thus, in the case of a non-robust boundary equilibrium (and in similar cases), we obtain an additional local necessary condition, but this condition is somewhat weaker than the one obtained for interior equilibria.

To understand this result, note that for the player that chooses an interior strategy, the sign of the payoff differential along the boundary segment shown in Fig. 3 will be determined by the third (or even higher) derivative of her payoff function. Furthermore, while the proofs of Propositions 3 and 4 differ, the two results are intuitively closely related. Specifically, the third-order condition in Proposition 4 captures a local monotonicity condition that needs to be satisfied to disallow the local construction of an ordinal potential as done in the proof of Proposition 3.

7 Applications

In this section, we review two classes of games traditionally studied by economists, and observe that, for these games, our necessary conditions are also sufficient.²²

7.1 Contests

In the n -player *lottery contest* with valuations $V_1 > 0, \dots, V_n > 0$, player i 's payoff is given by

$$u_i(x_1, \dots, x_n) = \begin{cases} \frac{x_i}{x_1 + \dots + x_n} V_i - x_i, & \text{if } x_1 + \dots + x_n > 0 \\ \frac{1}{n} V_i & \text{if } x_1 + \dots + x_n = 0, \end{cases} \quad (23)$$

where we assume that $X_1 = \dots = X_n = [0, V^{\max}]$, with $V^{\max} = \max_{i \in N} V_i$. It follows from a general result of Szidarovszky and Okuguchi (1997) that this game has a unique (yet not necessarily interior) Cournot–Nash equilibrium $x_N^* = (x_1^*, \dots, x_n^*)$.

Rather than applying our criterion to the n -player equilibrium, we will consider an equilibrium in the two-player game between arbitrary players $i, j \in N$ with $j \neq i$, assuming that all remaining players remain inactive. So let $x_N^\# = (x_i^\#, x_j^\#, x_{-i,j}^\#) \in X_N$ be such that the following conditions hold:

$$x_i^\# > 0, x_j^\# > 0, x_{-i,j}^\# = (0, \dots, 0) \in \mathbb{R}^{n-2}, \text{ and} \quad (24)$$

$$\frac{\partial u_i(x_N^\#)}{\partial x_i} = \frac{\partial u_j(x_N^\#)}{\partial x_j} = 0. \quad (25)$$

²² Additional illustrations can be found in the working paper (Ewerhart 2017).

In the bilateral game between players i and j , equilibrium efforts are given by the well-known expressions (cf. Konrad 2009)

$$x_i^\# = \frac{V_i^2 V_j}{(V_i + V_j)^2} \quad \text{and} \quad x_j^\# = \frac{V_i V_j^2}{(V_i + V_j)^2}. \quad (26)$$

The cross-derivative for player i is given by

$$\frac{\partial^2 u_i(x_N^\#)}{\partial x_j \partial x_i} = \frac{x_i^\# - x_j^\#}{(x_i^\# + x_j^\#)^3} V_i = \frac{V_i^2 - V_j^2}{(V_i V_j)^2}. \quad (27)$$

An analogous expression may be derived for player j . We therefore see that the necessary condition stated as Proposition 1 holds if and only if $V_i = V_j$. On the other hand, if $V_1 = \dots = V_n$, then the lottery contest can be shown to admit a generalized ordinal potential (Ewerhart and Kukushkin 2019). Thus, in the case of the lottery contest with heterogeneous valuations, our criterion is indeed not only necessary but also sufficient.²³

7.2 Differentiated Bertrand competition

There are $n \geq 2$ firms $i = 1, \dots, n$ with differentiated products. Suppose that each firm $i \in N$ chooses a price $p_i \geq 0$, and subsequently sells a quantity

$$q_i(p_N) = Q_i - s_i p_i + \sum_{j \neq i} \theta_{ij} p_j, \quad (28)$$

where $Q_i > 0$, $s_i > 0$, and $\theta_{ij} \neq 0$ are arbitrary parameters.²⁴ Firm i 's cost function is given by $C_i(q_i) = c_i q_i^2$, where $c_i > 0$. Thus, firm i 's profit reads

$$u_i(p_N) = p_i q_i(p_N) - C_i(q_i(p_N)) \quad (i \in N). \quad (29)$$

Suppose that there exists a price vector $p_N^* > 0$ at which marginal profits vanish (and at which profits are positive), for all firms.²⁵ Cross-derivatives are given by

²³ This example illustrates likewise the local nature of our conditions. Indeed, for $x_i \neq x_j$, the product of the cross-derivatives is negative in the lottery contest.

²⁴ If $\theta_{ij} = 0$ for some $i \neq j$, then the necessary and sufficient conditions (31, 32) identified below must be complemented by additional conditions analogous to (32) for any number $m \in \{4, \dots, n\}$ of players. Moreover, the functional form of the potential may differ somewhat from (33). In a nutshell, one first notes that, for a generalized ordinal potential to exist in the Bertrand game, it is necessary that, for any $i, j \in N$ with $i \neq j$, it holds that $\theta_{ij} = 0$ if and only if $\theta_{ji} = 0$. Then, in any connected component of the directed weighted graph defined by the bilateral price externalities $\{\theta_{ij}\}$, one chooses a spanning tree and defines a potential weight for a firm (or node) analogous to (35) by taking the product over all absolute price externalities represented as weights in the tree directed toward the firm. The details are omitted.

²⁵ This is the case, for instance, if $\max_{j \neq i} |\theta_{ij}|$ is sufficiently small.

$$\frac{\partial^2 u_i(p_N^*)}{\partial p_j \partial p_i} = \theta_{ij}(1 + 2s_i c_i) \quad (i, j \in N \text{ s.t. } j \neq i) \quad (30)$$

Note that $1 + 2s_i c_i > 0$ for $i \in N$. Therefore, from Theorem 1, necessary for the existence of a generalized ordinal potential is that²⁶

$$\{i, j \in N \text{ s.t. } i \neq j\} \Rightarrow \text{sgn}(\theta_{ij}) = \text{sgn}(\theta_{ji}), \quad (31)$$

and

$$\{i, j, k \in N \text{ s.t. } i \neq j \neq k \neq i\} \Rightarrow \theta_{ij}\theta_{jk}\theta_{ki} = \theta_{ji}\theta_{kj}\theta_{ik}. \quad (32)$$

Conversely, we have the following observation.

Proposition 5 *Suppose that conditions (31) and (32) hold. Then,*

$$P(p_N) = \sum_{i=1}^n \hat{w}_i p_i \left\{ Q_i \left(1 - \frac{p_i}{2p_i^0} \right) + \sum_{j=1}^{i-1} \theta_{ij} p_j \right\}, \quad (33)$$

with

$$p_i^0 = \frac{(1 + 2s_i c_i) Q_i}{2s_i(1 + s_i c_i)} \quad (i \in N), \quad \text{and} \quad (34)$$

$$\hat{w}_i = (|\theta_{12}| \cdot \dots \cdot |\theta_{i-1,i}|) \cdot (|\theta_{i+1,i}| \cdot \dots \cdot |\theta_{nn-1}|) \quad (i \in N), \quad (35)$$

is a generalized ordinal potential for the differentiated Bertrand game.

Thus, also in this class of games, our conditions turn out to both necessary and sufficient for the existence of a generalized ordinal potential.²⁷

8 Concluding remarks

In this paper, we have identified conditions necessary for the existence of a generalized ordinal potential in any game with interval strategy spaces and twice continuously differentiable payoff functions. For selected classes of games, including lottery contests with heterogeneous valuations, and differentiated Bertrand games with linear demand and quadratic costs, these conditions are also sufficient. Thus, a first step toward the differentiable characterization of ordinal and generalized ordinal potential games has been made.

²⁶ As usual, $\text{sgn}(\cdot)$ denotes the *sign function*, with $\text{sgn}(z) = 1$ if $z > 0$, $\text{sgn}(z) = 0$ if $z = 0$, and $\text{sgn}(z) = -1$ if $z < 0$.

²⁷ In addition, the example shows that our necessary conditions may indeed be indicative of sufficient conditions.

Further extensions are possible. For example, using similar methods, a necessary differentiable condition may be obtained from the analysis of non-local strict improvement cycles. This idea is developed in the working paper version (Ewerhart 2017). There as well, we extend Propositions 1 and 2 to games with multi-dimensional strategy spaces. However, given the lack of a suitable generalization of the notion of semipositivity, it has to remain an open question if the local analogue of the Monderer–Shapley condition extends accordingly.²⁸

Echenique (2004) has shown that, generically, a finite two-player ordinal potential game is a game of strategic complements (potentially after reordering strategic spaces), yet an ordinal potential game with more than two players need not generically be a game of strategic complements. While it is not straightforward to compare the two settings, some intuitions carry over. In particular, our findings provoke the question if any finite ordinal potential game with more than two players can be turned, by reordering of strategy spaces, into a game of pairwise strategic complements and substitutes. We do not know the answer to this question.

Any smooth exact potential game with negative definite Hessian is a concave game in the sense of Rosen (1965).²⁹ The converse, however, is not true in general. In fact, our results imply that a concave game need not even admit a generalized ordinal potential. Indeed, the lottery contest is concave (Ewerhart and Quartieri 2019), but as seen above, it does not admit a generalized ordinal potential unless valuations are homogeneous.

There are three natural classes of smooth games that satisfy our most stringent necessary conditions. These are symmetric two-player zero-sum games, supermodular sum-aggregative game, and symmetric games that admit only symmetric equilibria.³⁰ To the extent that necessary conditions are indicative of sufficient conditions, one might speculate that these classes of games, potentially under additional technical conditions, have the property that an ordinal potential can be constructed locally.³¹

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Appendix: Proofs

Proof of Proposition 1 By contradiction. Suppose that, at some profile $x_N^* \in X_N$, and for some players i and j with $i \neq j$, we have

²⁸ A multi-dimensional variant of the Monderer–Shapley condition for exact potential games can be found in Deb (2008, 2009).

²⁹ See Neyman (1997), Ui (2008), and Hofbauer and Sandholm (2009).

³⁰ Smooth symmetric games admit at least one symmetric equilibrium under standard assumptions (Moulin 1986, p. 115). However, there are also large classes of economically relevant symmetric games that admit only asymmetric pure-strategy Nash equilibria (cf. Amir et al. 2010).

³¹ The author is presently exploring the validity of these conjectures.

$$x_i^* \in \mathring{X}_i, \quad x_j^* \in \mathring{X}_j, \quad \frac{\partial u_i(x_N^*)}{\partial x_i} = \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \quad \text{and} \quad (36)$$

$$\frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} < 0. \quad (37)$$

By renaming players, if necessary, we may assume w.l.o.g. that

$$\frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} > 0 > \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j}, \quad (38)$$

which corresponds to the case shown in Fig. 1. It is claimed now that, for any sufficiently small $\varepsilon > 0$, the payoff difference corresponding to the upper side of the square satisfies

$$\Delta_i^+(\varepsilon) \equiv u_i(x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-ij}^*) - u_i(x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-ij}^*) > 0. \quad (39)$$

To prove this, we determine the second-order Taylor approximation of $\Delta_i^+(\varepsilon)$ at $\varepsilon = 0$. Writing $f(\varepsilon)$ for $\Delta_i^+(\varepsilon)$, our differentiability assumptions combined with Taylor's theorem imply that there is a remainder term $r(\cdot)$ with $\lim_{\varepsilon \rightarrow 0} r(\varepsilon) = 0$ such that for any sufficiently small $\varepsilon > 0$,

$$f(\varepsilon) = f(0) + f'(0)\varepsilon + \frac{1}{2}f''(0)\varepsilon^2 + r(\varepsilon)\varepsilon^2. \quad (40)$$

Clearly, $f(0) = 0$. As for the first derivative $f'(0)$, one obtains

$$\begin{aligned} \frac{\partial \Delta_i^+(\varepsilon)}{\partial \varepsilon} = & \left\{ \frac{\partial u_i(x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-ij}^*)}{\partial x_i} + \frac{\partial u_i(x_i^* + \varepsilon, x_j^* + \varepsilon, x_{-ij}^*)}{\partial x_j} \right\} \\ & - \left\{ -\frac{\partial u_i(x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-ij}^*)}{\partial x_i} + \frac{\partial u_i(x_i^* - \varepsilon, x_j^* + \varepsilon, x_{-ij}^*)}{\partial x_j} \right\}. \end{aligned} \quad (41)$$

Evaluating at $\varepsilon = 0$, and subsequently exploiting the necessary first-order condition for player i at the interior equilibrium x_N^* , we find

$$f'(0) = \frac{\partial \Delta_i^+(0)}{\partial \varepsilon} = 2 \cdot \frac{\partial u_i(x_N^*)}{\partial x_i} = 0. \quad (42)$$

Next, consider the second derivative of $\Delta_i^+(\varepsilon)$ at $\varepsilon = 0$, i.e.,

$$\frac{\partial^2 \Delta_i^+(0)}{\partial \varepsilon^2} = \left\{ \frac{\partial^2 u_i(x_N^*)}{\partial x_i^2} + \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} + \frac{\partial^2 u_i(x_N^*)}{\partial x_i \partial x_j} + \frac{\partial^2 u_i(x_N^*)}{\partial x_j^2} \right\} \quad (43)$$

$$\begin{aligned}
& - \left\{ \frac{\partial^2 u_i(x_N^*)}{\partial x_i^2} - \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} - \frac{\partial^2 u_i(x_N^*)}{\partial x_i \partial x_j} + \frac{\partial^2 u_i(x_N^*)}{\partial x_j^2} \right\} \\
& = 2 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} + 2 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_i \partial x_j}.
\end{aligned} \tag{44}$$

Invoking Schwarz's theorem regarding the equality of cross-derivatives for twice continuously differentiable functions, and subsequently using (38), one obtains

$$f''(0) = \frac{\partial^2 \Delta_i^+(0)}{\partial \varepsilon^2} = 4 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} > 0. \tag{45}$$

In sum, we have shown that $f'(0) = f(0) = 0$ and $f''(0) > 0$. Thus, using (40), it follows that $\Delta_i^+(\varepsilon) > 0$ for any sufficiently small $\varepsilon > 0$. Analogous arguments can be used to deal with the other three sides of the square. Specifically, one defines

$$\Delta_j^+(\varepsilon) = u_j(x_j^* - \varepsilon, x_i^* + \varepsilon, x_{-ij}^*) - u_j(x_j^* + \varepsilon, x_i^* + \varepsilon, x_{-ij}^*), \tag{46}$$

$$\Delta_i^-(\varepsilon) = u_i(x_i^* - \varepsilon, x_j^* - \varepsilon, x_{-ij}^*) - u_i(x_i^* + \varepsilon, x_j^* - \varepsilon, x_{-ij}^*), \tag{47}$$

$$\Delta_j^-(\varepsilon) = u_j(x_j^* + \varepsilon, x_i^* - \varepsilon, x_{-ij}^*) - u_j(x_j^* - \varepsilon, x_i^* - \varepsilon, x_{-ij}^*), \tag{48}$$

and now readily verifies that

$$\frac{\partial \Delta_j^+(0)}{\partial \varepsilon} = (-2) \cdot \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \tag{49}$$

$$\frac{\partial \Delta_i^-(0)}{\partial \varepsilon} = (-2) \cdot \frac{\partial u_i(x_N^*)}{\partial x_i} = 0, \tag{50}$$

$$\frac{\partial \Delta_j^-(0)}{\partial \varepsilon} = 2 \cdot \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \tag{51}$$

and that

$$\frac{\partial^2 \Delta_j^+(0)}{\partial \varepsilon^2} = (-4) \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} > 0, \tag{52}$$

$$\frac{\partial^2 \Delta_i^-(0)}{\partial \varepsilon^2} = 4 \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} > 0, \tag{53}$$

$$\frac{\partial^2 \Delta_j^-(0)}{\partial \varepsilon^2} = (-4) \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} > 0. \quad (54)$$

It follows that $\Delta_i^+(\varepsilon) > 0$, $\Delta_j^+(\varepsilon) > 0$, $\Delta_i^-(\varepsilon) > 0$, and $\Delta_j^-(\varepsilon) > 0$ all hold for $\varepsilon > 0$ small enough. But then, the finite sequence (6) is a strict improvement cycle, which is incompatible with the existence of a generalized ordinal potential by Lemma 1. \square

Proof of Proposition 2 Suppose that the modified Jacobian $J = J(x_N)$ is semipositive. Then, by definition, there exists a vector $\lambda_N = (\lambda_1, \dots, \lambda_n)^T \in \mathbb{R}^n$ with $\lambda_N > 0$ such that $J\lambda_N > 0$. Consider the finite sequence

$$\begin{aligned} \dots &\rightarrow x_N^{(1,+)}(\varepsilon) = (x_1^* + \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, \dots, x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \\ &\rightarrow x_N^{(2,+)}(\varepsilon) = (x_1^* + \lambda_1\varepsilon, x_2^* + \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, \dots, x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \rightarrow \\ &\vdots \\ &\rightarrow x_N^{(n,+)}(\varepsilon) = (x_1^* + \lambda_1\varepsilon, x_2^* + \lambda_2\varepsilon, x_3^* + \lambda_3\varepsilon, \dots, x_{n-1}^* + \lambda_{n-1}\varepsilon, x_n^* + \lambda_n\varepsilon) \\ &\rightarrow x_N^{(1,-)}(\varepsilon) = (x_1^* - \lambda_1\varepsilon, x_2^* + \lambda_2\varepsilon, x_3^* + \lambda_3\varepsilon, \dots, x_{n-1}^* + \lambda_{n-1}\varepsilon, x_n^* + \lambda_n\varepsilon) \\ &\rightarrow x_N^{(2,-)}(\varepsilon) = (x_1^* - \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* + \lambda_3\varepsilon, \dots, x_{n-1}^* + \lambda_{n-1}\varepsilon, x_n^* + \lambda_n\varepsilon) \rightarrow \\ &\vdots \\ &\rightarrow x_N^{(n,-)}(\varepsilon) = (x_1^* - \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, \dots, x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \rightarrow \dots, \end{aligned} \quad (55)$$

where $\varepsilon > 0$ is a small constant as before. Figure 2 illustrates this path for $n = 3$, where the rectangular-shaped box has sides of respective length $\varepsilon_i = \lambda_i\varepsilon$ for $i = 1, 2, 3$. It is claimed that, for any $\varepsilon > 0$ sufficiently small, the following four conditions hold:

1. player 1's payoff at $x_N^{(1,+)}(\varepsilon)$ is strictly higher than at $x_N^{(n,-)}(\varepsilon)$;
2. for $i = 2, \dots, n$, player i 's payoff at $x_N^{(i,+)}(\varepsilon)$ is strictly higher than at $x_N^{(i-1,+)}(\varepsilon)$;
3. player 1's payoff at $x_N^{(1,-)}(\varepsilon)$ is strictly higher than at $x_N^{(n,+)}(\varepsilon)$;
4. for $i = 2, \dots, n$, player i 's payoff at $x_N^{(i,-)}(\varepsilon)$ is strictly higher than at $x_N^{(i-1,-)}(\varepsilon)$.

To establish claim (1), we proceed as in the proof of Proposition 1, and consider the first two derivatives of the payoff difference

$$\Delta^{(1,+)}(\varepsilon) = u_1(x_N^{(1,+)}(\varepsilon)) - u_1(x_N^{(n,-)}(\varepsilon)) \quad (56)$$

$$\begin{aligned} &= u_1(x_1^* + \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, \dots, x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \\ &\quad - u_1(x_1^* - \lambda_1\varepsilon, x_2^* - \lambda_2\varepsilon, x_3^* - \lambda_3\varepsilon, \dots, x_{n-1}^* - \lambda_{n-1}\varepsilon, x_n^* - \lambda_n\varepsilon) \end{aligned} \quad (57)$$

at $\varepsilon = 0$. The first derivative of $\Delta^{(1,+)}(\varepsilon)$ at $\varepsilon = 0$ is given by

$$\begin{aligned} \frac{\partial \Delta^{(1,+)}(0)}{\partial \varepsilon} &= \left(\lambda_1 \frac{\partial u_1(x_N^*)}{\partial x_1} - \lambda_2 \frac{\partial u_1(x_N^*)}{\partial x_2} - \dots - \lambda_n \frac{\partial u_1(x_N^*)}{\partial x_n} \right) \\ &\quad - \left(-\lambda_1 \frac{\partial u_1(x_N^*)}{\partial x_1} - \lambda_2 \frac{\partial u_1(x_N^*)}{\partial x_2} - \dots - \lambda_n \frac{\partial u_1(x_N^*)}{\partial x_n} \right) \end{aligned} \quad (58)$$

$$= 2\lambda_1 \frac{\partial u_1(x_N^*)}{\partial x_1}. \quad (59)$$

Hence, from player 1's first-order condition,

$$\frac{\partial \Delta^{(1,+)}(0)}{\partial \varepsilon} = 0. \quad (60)$$

Next, one considers the second derivative of $\Delta^{(1,+)}(\varepsilon)$ at $\varepsilon = 0$, i.e.,

$$\begin{aligned} \frac{\partial^2 \Delta^{(1,+)}(0)}{\partial \varepsilon^2} &= \left\{ (\lambda_1)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_1^2} - \lambda_2 \lambda_1 \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} - \dots - \lambda_n \lambda_1 \frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_1} \right. \\ &\quad - \lambda_1 \lambda_2 \frac{\partial^2 u_1(x_N^*)}{\partial x_1 \partial x_2} + (\lambda_2)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_2^2} + \dots + \lambda_n \lambda_2 \frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_2} \\ &\quad \vdots \\ &\quad \left. - \lambda_1 \lambda_n \frac{\partial^2 u_1(x_N^*)}{\partial x_1 \partial x_n} + \lambda_2 \lambda_n \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_n} + \dots + (\lambda_n)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_n^2} \right\} \\ &\quad - \left\{ (\lambda_1)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_1^2} + \lambda_2 \lambda_1 \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} + \dots + \lambda_n \lambda_1 \frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_1} \right. \\ &\quad + \lambda_1 \lambda_2 \frac{\partial^2 u_1(x_N^*)}{\partial x_1 \partial x_2} + (\lambda_2)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_2^2} + \dots + \lambda_n \lambda_2 \frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_2} \\ &\quad \vdots \\ &\quad \left. + \lambda_1 \lambda_n \frac{\partial^2 u_1(x_N^*)}{\partial x_1 \partial x_n} + \lambda_2 \lambda_n \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_n} + \dots + (\lambda_n)^2 \frac{\partial^2 u_1(x_N^*)}{\partial x_n^2} \right\}. \end{aligned} \quad (61)$$

Collecting terms, one obtains

$$\frac{\partial^2 \Delta^{(1,+)}(0)}{\partial \varepsilon^2} = 2\lambda_1 \left\{ -\lambda_2 \frac{\partial^2 u_1(x_N^*)}{\partial x_2 \partial x_1} - \lambda_3 \frac{\partial^2 u_1(x_N^*)}{\partial x_3 \partial x_1} - \dots - \lambda_n \frac{\partial^2 u_1(x_N^*)}{\partial x_n \partial x_1} \right\}. \quad (62)$$

Thus, using $\lambda_1 > 0$, and noting that the expression in the curly brackets corresponds to the first component of the vector $J\lambda_N > 0$, one arrives at

$$\frac{\partial^2 \Delta^{(1,+)}(0)}{\partial \varepsilon^2} > 0. \quad (63)$$

It follows that $\Delta^{(1,+)}(\varepsilon) > 0$ for any $\varepsilon > 0$ sufficiently small, which proves claim (1). To verify claims (2) through (4), define payoff differences

$$\Delta^{(i,+)}(\varepsilon) = u_i(x_N^{(i,+)}(\varepsilon)) - u_i(x_N^{(i-1,+)}(\varepsilon)) \quad (i = 2, \dots, n), \quad (64)$$

$$\Delta^{(1,-)}(\varepsilon) = u_1(x_N^{(1,-)}(\varepsilon)) - u_1(x_N^{(n,+)}(\varepsilon)), \quad (65)$$

$$\Delta^{(i,-)}(\varepsilon) = u_i(x_N^{(i,-)}(\varepsilon)) - u_i(x_N^{(i-1,+)}(\varepsilon)) \quad (i = 2, \dots, n). \quad (66)$$

Using players' necessary first-order conditions, it is straightforward to validate that

$$\frac{\partial \Delta^{(i,+)}(0)}{\partial \varepsilon} = 0 \quad (i = 2, \dots, n), \quad (67)$$

$$\frac{\partial \Delta^{(1,-)}(0)}{\partial \varepsilon} = 0, \quad (68)$$

$$\frac{\partial \Delta^{(i,-)}(0)}{\partial \varepsilon} = 0 \quad (i = 2, \dots, n). \quad (69)$$

Moreover, for $i = 2, \dots, n$, calculations analogous to (61, 62) yield

$$\frac{\partial^2 \Delta^{(i,+)}(0)}{\partial \varepsilon^2} = 2\lambda_i \left\{ \lambda_1 \frac{\partial^2 u_i(x_N^*)}{\partial x_1 \partial x_i} + \dots + \lambda_{i-1} \frac{\partial^2 u_i(x_N^*)}{\partial x_{i-1} \partial x_i} \right. \quad (70)$$

$$\left. - \lambda_{i+1} \frac{\partial^2 u_i(x_N^*)}{\partial x_{i+1} \partial x_i} - \dots - \lambda_n \frac{\partial^2 u_i(x_N^*)}{\partial x_n \partial x_i} \right\} \quad (71)$$

$$> 0,$$

where the expression in the curly brackets corresponds to the i 's component of the vector $J\lambda_N > 0$. Finally, one notes that, from $d(-\varepsilon)^2 = d\varepsilon^2$, it follows that

$$\frac{\partial^2 \Delta^{(i,-)}(0)}{\partial \varepsilon^2} = \frac{\partial^2 \Delta^{(i,+)}(0)}{\partial \varepsilon^2} \quad (i = 1, \dots, n). \quad (72)$$

In sum, this proves claims (2) through (4). Thus, there exists a strict improvement cycle in the generalized ordinal potential game Γ . Since this is impossible, the proposition follows. \square

The three lemmas below will be used in the proof of Theorem 1. Following the literature, we will call a square matrix $A \in \mathbb{R}^{n \times n}$ *inverse nonnegative* if the matrix

inverse A^{-1} exists and if, in addition, all entries of A^{-1} are nonnegative. The following lemma provides a useful recursive characterization of semipositivity.

Lemma 2 (Johnson et al. 1994). *A square matrix $A \in \mathbb{R}^{n \times n}$ is semipositive if and only if at least one of the following two conditions holds:*

1. *A is inverse nonnegative;*
2. *there exists $m \in \{1, \dots, n-1\}$ and a submatrix $\hat{A} \in \mathbb{R}^{n \times m}$ obtained from A via deletion of $n-m$ columns, such that all $m \times m$ submatrices of \hat{A} are semipositive.*

Using the lemma above, we may derive the following implication of Proposition 2.

Lemma 3 *Suppose that the smooth n -player game Γ admits a generalized ordinal potential. Then, at any profile x_N^* and for any set $\{i, j, k\} \subseteq N$ of pairwise different players,*

$$\left\{ x_i^* \in \overset{\circ}{X}_i, x_j^* \in \overset{\circ}{X}_j, x_k^* \in \overset{\circ}{X}_k, \text{ and } \frac{\partial u_i(x_N^*)}{\partial x_i} = \frac{\partial u_j(x_N^*)}{\partial x_j} = \frac{\partial u_k(x_N^*)}{\partial x_k} = 0 \right\} \\ \Rightarrow \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_k \partial x_j} \cdot \frac{\partial^2 u_k(x_N^*)}{\partial x_i \partial x_k} = \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} \cdot \frac{\partial^2 u_k(x_N^*)}{\partial x_j \partial x_k} \cdot \frac{\partial^2 u_i(x_N^*)}{\partial x_k \partial x_i}. \quad (73)$$

Proof Fix some profile x_N^* and a set $\{i, j, k\} \subseteq N$ of pairwise different players such that

$$x_i^* \in \overset{\circ}{X}_i, \quad x_j^* \in \overset{\circ}{X}_j, \quad x_k^* \in \overset{\circ}{X}_k, \quad \text{and} \quad (74)$$

$$\frac{\partial u_i(x_N^*)}{\partial x_i} = \frac{\partial u_j(x_N^*)}{\partial x_j} = \frac{\partial u_k(x_N^*)}{\partial x_k} = 0. \quad (75)$$

Using the notation

$$\chi_{ij} = \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i}, \quad \chi_{jk} = \frac{\partial^2 u_j(x_N^*)}{\partial x_k \partial x_j}, \dots \quad (76)$$

for cross-derivatives, it needs to be shown that

$$\chi_{ij} \cdot \chi_{jk} \cdot \chi_{ki} = \chi_{ji} \cdot \chi_{kj} \cdot \chi_{ik}. \quad (77)$$

Suppose first that Γ exhibits weak strategic complements at x_N^* , i.e.,

$$\chi_{ij} \geq 0, \quad \chi_{ji} \geq 0, \quad \chi_{ik} \geq 0, \quad \chi_{ki} \geq 0, \quad \chi_{jk} \geq 0, \quad \text{and} \quad \chi_{kj} \geq 0. \quad (78)$$

Consider now a small circular path along the edges of a small three-dimensional rectangular-shaped box around x_N^* . Along the path, players i, j , and k move in this order, with i and k initially increasing their strategies, while j initially decreases her strategy. Since this corresponds to flipping around player j 's strategy space, all

cross-derivatives involving player j change sign, so that the corresponding modified Jacobian reads

$$J_3 = \begin{pmatrix} 0 & \chi_{ij} & -\chi_{ik} \\ -\chi_{ji} & 0 & \chi_{jk} \\ \chi_{ki} & -\chi_{kj} & 0 \end{pmatrix}. \quad (79)$$

By Proposition 2, J_3 cannot be inverse nonnegative. To prove (77), it suffices to show that the determinant of J_3 ,

$$|J_3| = \chi_{ij}\chi_{jk}\chi_{ki} - \chi_{ji}\chi_{kj}\chi_{ik}, \quad (80)$$

vanishes. To provoke a contradiction, suppose first that $|J_3| > 0$. Then, from weak strategic complements at x_N^* , all the entries of the matrix inverse of J_3 ,

$$(J_3)^{-1} = \frac{1}{|J_3|} \begin{pmatrix} \chi_{jk}\chi_{kj} & \chi_{ik}\chi_{kj} & \chi_{ij}\chi_{jk} \\ \chi_{ki}\chi_{jk} & \chi_{ik}\chi_{ki} & \chi_{ji}\chi_{ik} \\ \chi_{ji}\chi_{kj} & \chi_{ij}\chi_{ki} & \chi_{ij}\chi_{ji} \end{pmatrix}, \quad (81)$$

are nonnegative, in contradiction to the fact that J_3 is not inverse nonnegative. Hence, $|J_3| \leq 0$. Suppose next that $|J_3| < 0$. Then, by running through the above-considered path in the opposite direction (i.e., by exchanging the roles of players i and k), Proposition 2 implies that

$$\hat{J}_3 = \begin{pmatrix} 0 & \chi_{kj} & -\chi_{ki} \\ -\chi_{jk} & 0 & \chi_{ji} \\ \chi_{ik} & -\chi_{ij} & 0 \end{pmatrix} \quad (82)$$

is not inverse nonnegative. Clearly, the determinant of \hat{J}_3 is

$$|\hat{J}_3| = \chi_{kj}\chi_{ji}\chi_{ik} - \chi_{jk}\chi_{ij}\chi_{ki} = -|J_3| > 0. \quad (83)$$

Hence, using an expression for the matrix inverse analogous to (81), all entries of $(\hat{J}_3)^{-1}$ are seen to be nonnegative, in contradiction to the fact that \hat{J}_3 is not inverse nonnegative. It follows that $|J_3| = 0$, which proves the claim in the case where Γ exhibits weak strategic complements at x_N^* . The case of weak strategic substitutes, where

$$\chi_{ij} \leq 0, \quad \chi_{ji} \leq 0, \quad \chi_{ik} \leq 0, \quad \chi_{ki} \leq 0, \quad \chi_{jk} \leq 0, \quad \text{and} \quad \chi_{kj} \leq 0, \quad (84)$$

is entirely analogous, and therefore omitted. We are now in the position to address the general case. From Proposition 1, we know that Γ exhibits pairwise weak strategic complements or substitutes at x_N^* . Hence, up to a renaming of the players, there are only two remaining cases:

1. Weak strategic complements at x_N^* between player i and each of players j and k , as well as strategic substitutes at x_N^* between players j and k ;

2. Weak strategic substitutes at x_N^* between player i and each of players j and k , as well as weak strategic complements at x_N^* between players j and k .

In either case, by flipping around the strategy space of player i , the game may be transformed into a game that exhibits either weak strategic substitutes at x_N^* or weak strategic complements at x_N^* . Since the operation of flipping around individual strategy spaces does not affect the validity of Eq. (77), we find that the conclusion indeed holds in the general case. \square

Lemma 4 Suppose that the smooth n -player game Γ admits a generalized ordinal potential. Then, at any interior strategy profile x_N^* at which all first-order conditions hold, and for any set of pairwise distinct players $\{i_1, \dots, i_m\} \subseteq N$ with $m \geq 3$, using the notation introduced in (76), it holds that

$$\chi_{i_1 i_2} \cdot \chi_{i_2 i_3} \cdot \dots \cdot \chi_{i_{m-1} i_m} \cdot \chi_{i_m i_1} = \chi_{i_2 i_1} \cdot \chi_{i_3 i_2} \cdot \dots \cdot \chi_{i_m i_{m-1}} \cdot \chi_{i_1 i_m}, \quad (85)$$

provided that $\chi_{i_1 i_3} \cdot \chi_{i_3 i_1} \neq 0, \dots, \chi_{i_1 i_{m-1}} \cdot \chi_{i_{m-1} i_1} \neq 0$.

Proof The proof proceeds by induction. The case $m = 3$ follows directly from Lemma 3. Suppose that $m \geq 4$, and let $\{i_1, i_2, \dots, i_m\}$ be an arbitrary set of pairwise different players. Suppose that the claim has been shown for any $m' \in \{3, 4, \dots, m-1\}$. Then, a consideration of the two subsets $\{i_1, i_2, \dots, i_{m-1}\}$ and $\{i_1, i_{m-1}, i_m\} = \{i_{m-1}, i_m, i_1\}$ shows that

$$\chi_{i_1 i_2} \cdot \dots \cdot \chi_{i_{m-2} i_{m-1}} \cdot \chi_{i_{m-1} i_1} = \chi_{i_2 i_1} \cdot \dots \cdot \chi_{i_{m-1} i_{m-2}} \cdot \chi_{i_1 i_{m-1}}, \quad \text{and} \quad (86)$$

$$\chi_{i_{m-1} i_m} \cdot \chi_{i_m i_1} \cdot \chi_{i_1 i_{m-1}} = \chi_{i_m i_{m-1}} \cdot \chi_{i_1 i_m} \cdot \chi_{i_{m-1} i_1}. \quad (87)$$

Combining the two equations via multiplication yields

$$\begin{aligned} & (\chi_{i_1 i_2} \cdot \dots \cdot \chi_{i_{m-2} i_{m-1}} \cdot \chi_{i_{m-1} i_1}) \cdot (\chi_{i_{m-1} i_m} \cdot \chi_{i_m i_1} \cdot \chi_{i_1 i_{m-1}}) \\ &= (\chi_{i_2 i_1} \cdot \dots \cdot \chi_{i_{m-1} i_{m-2}} \cdot \chi_{i_1 i_{m-1}}) \cdot (\chi_{i_m i_{m-1}} \cdot \chi_{i_1 i_m} \cdot \chi_{i_{m-1} i_1}). \end{aligned} \quad (88)$$

By assumption, $\chi_{i_1 i_{m-1}} \cdot \chi_{i_{m-1} i_1} \neq 0$. Hence, eliminating these common nonzero factors, (88) implies

$$\chi_{i_1 i_2} \cdot \dots \cdot \chi_{i_{m-1} i_m} \cdot \chi_{i_m i_1} = \chi_{i_2 i_1} \cdot \dots \cdot \chi_{i_m i_{m-1}} \cdot \chi_{i_1 i_m}, \quad (89)$$

as claimed. This concludes the induction step, and therefore proves the lemma. \square

Proof of Theorem 1 Let x_N^* be an interior strategy profile such that all first-order conditions hold at x_N^* and such that $\chi_{ij} \neq 0$ for all $i \neq j$. We need to find positive constants $w_1 > 0, \dots, w_n > 0$ such that

$$\chi_{ij} w_i = \chi_{ji} w_j \quad (i, j \in N, j \neq i). \quad (90)$$

It is claimed that

$$w_i = (|\chi_{12}| \cdot \dots \cdot |\chi_{i-1i}|) \cdot (|\chi_{i+1i}| \cdot \dots \cdot |\chi_{nn-1}|) \quad (i \in N) \quad (91)$$

does the job, with the usual convention that any empty product equals one. Clearly, it suffices to check (90) for $i < j$, because if $i > j$, one may just exchange the two sides of Eq. (90). Suppose first that $n = 2$. Then, (91) implies $w_1 = |\chi_{21}|$ and $w_2 = |\chi_{12}|$, and the claim follows directly from Proposition 1. Suppose next that $n \geq 3$. Splitting the product in the second bracket of (91), one obtains

$$\begin{aligned} \chi_{ij} w_i &= (|\chi_{12}| \cdot \dots \cdot |\chi_{i-1i}|) \\ &\quad \cdot \operatorname{sgn}(\chi_{ij}) \cdot (|\chi_{i+1i}| \cdot \dots \cdot |\chi_{jj-1}| \cdot |\chi_{ij}|) \\ &\quad \cdot (|\chi_{j+1j}| \cdot \dots \cdot |\chi_{nn-1}|). \end{aligned} \quad (92)$$

From Proposition 1,

$$\operatorname{sgn}(\chi_{ij}) = \operatorname{sgn}(\chi_{ji}). \quad (93)$$

Moreover, from Lemma 4,

$$\chi_{i+1i} \cdot \dots \cdot \chi_{jj-1} \cdot \chi_{ij} = \chi_{ii+1} \cdot \dots \cdot \chi_{j-1j} \cdot \chi_{ji}. \quad (94)$$

Hence, using (93) and (94) in relationship (92) delivers

$$\chi_{ij} w_i = (|\chi_{12}| \cdot \dots \cdot |\chi_{i-1i}|) \quad (95)$$

$$\cdot \operatorname{sgn}(\chi_{ji}) \cdot (|\chi_{ii+1}| \cdot \dots \cdot |\chi_{j-1j}| \cdot |\chi_{ji}|)$$

$$\cdot (|\chi_{j+1j}| \cdot \dots \cdot |\chi_{nn-1}|)$$

$$= \chi_{ji} \cdot (|\chi_{12}| \cdot \dots \cdot |\chi_{j-1j}|) \cdot (|\chi_{j+1j}| \cdot \dots \cdot |\chi_{nn-1}|) \quad (96)$$

$$= \chi_{ji} w_j. \quad (97)$$

This proves the claim and, hence, the theorem. \square

Proof of Proposition 3 To construct an ordinal potential locally at x_N^* , start by letting

$$S_j \equiv \operatorname{sgn}\left(\frac{\partial u_j(x_N^*)}{\partial x_j}\right) \in \{1, -1\} \quad (j \in N \setminus \{i\}), \quad (98)$$

where the sign function $\operatorname{sgn}(\cdot)$ is defined in Sect. 7. Next, choose a constant b_j , for each $j \in N \setminus \{i\}$, such that $|\partial u_i(x_N^*) / \partial x_j| < b_j$. Then, since Γ is smooth, there is a small rectangular neighborhood $U_N(x_N^*)$ of x_N^* such that

$$U_N(x_N^*) \subseteq \left\{ x_N \in X_N \text{ s.t. } \operatorname{sgn} \left(\frac{\partial u_j(x_N)}{\partial x_j} \right) = S_j \text{ and } \left| \frac{\partial u_i(x_N)}{\partial x_j} \right| < b_j \text{ for all } j \neq i \right\}. \quad (99)$$

We claim that

$$P(x_N) = u_i(x_N) + \sum_{j \in N \setminus \{i\}} S_j b_j x_j \quad (100)$$

is an ordinal potential for Γ when joint strategy choices are restricted to $U_N(x_N^*)$. To see this, note first that

$$P(x_i, x_{-i}) - P(\hat{x}_i, x_{-i}) = u_i(x_i, x_{-i}) - u_i(\hat{x}_i, x_{-i}), \quad (101)$$

for any $x_i \in X_i$, $\hat{x}_i \in X_i$, and $x_{-i} \in X_{-i}$. Next, let $j \in N \setminus \{i\}$. Take any $x_j \in X_j$, $\hat{x}_j \in X_j$, and $x_{-j} \in X_{-j}$ such that $(x_j, x_{-j}) \in U_N(x_N^*)$ and $(\hat{x}_j, x_{-j}) \in U_N(x_N^*)$. Then, from (100),

$$P(x_j, x_{-j}) - P(\hat{x}_j, x_{-j}) = u_i(x_j, x_{-j}) - u_i(\hat{x}_j, x_{-j}) + b_j S_j (x_j - \hat{x}_j). \quad (102)$$

Invoking the mean value theorem, there exists a strict convex combination \tilde{x}_j of \hat{x}_j and x_j such that

$$u_i(x_j, x_{-j}) - u_i(\hat{x}_j, x_{-j}) = \frac{\partial u_i(\tilde{x}_j, x_{-j})}{\partial x_j} (x_j - \hat{x}_j). \quad (103)$$

Hence,

$$P(x_j, x_{-j}) - P(\hat{x}_j, x_{-j}) = \left\{ \frac{\partial u_i(\tilde{x}_j, x_{-j})}{\partial x_j} + b_j S_j \right\} (x_j - \hat{x}_j). \quad (104)$$

To check the definition of an ordinal potential, suppose that $u_j(x_j, x_{-j}) - u_j(\hat{x}_j, x_{-j}) > 0$. Then, using (99), $S_j(x_j - \hat{x}_j) > 0$. Moreover, since $U_N(x_N^*)$ is convex, $|\partial u_i(\tilde{x}_j, x_{-j})/\partial x_j| < b_j$. It follows that $P(x_j, x_{-j}) - P(\hat{x}_j, x_{-j}) > 0$. Since these steps may be reversed, P is indeed an ordinal potential for Γ in the neighborhood $U_N(x_N^*)$. This proves the proposition. \square

Proof of Proposition 4 We adapt the proof of Proposition 1. To provoke a contradiction, suppose that

$$x_i^* = \underline{x}_i, x_j^* \in \hat{X}_j, \quad \frac{\partial u_i(x_N^*)}{\partial x_i} = \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \quad (105)$$

$$\frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} \cdot \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} < 0, \quad \text{and} \quad (106)$$

$$\frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} \cdot \frac{\partial^3 u_j(x_N^*)}{\partial x_j^3} < 0. \quad (107)$$

There are two cases. Suppose first that, as illustrated in Fig. 3,

$$\frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} > 0 > \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j}. \quad (108)$$

Then, condition (107) implies $\partial^3 u_j(x_N^*)/\partial x_j^3 > 0$. Given constants $\lambda_i > 0$ and $\lambda_j > 0$, let

$$\hat{\Delta}_i^+(\varepsilon) = u_i(x_i^* + \lambda_i \varepsilon, x_j^* + \lambda_j \varepsilon, x_{-ij}^*) - u_i(x_i^*, x_j^* + \lambda_j \varepsilon, x_{-ij}^*), \quad (109)$$

$$\hat{\Delta}_j^+(\varepsilon) = u_j(x_j^* - \lambda_j \varepsilon, x_i^* + \lambda_i \varepsilon, x_{-ij}^*) - u_j(x_j^* + \lambda_j \varepsilon, x_i^* + \lambda_i \varepsilon, x_{-ij}^*), \quad (110)$$

$$\hat{\Delta}_i^-(\varepsilon) = u_i(x_i^*, x_j^* - \lambda_j \varepsilon, x_{-ij}^*) - u_i(x_i^* + \lambda_i \varepsilon, x_j^* - \lambda_j \varepsilon, x_{-ij}^*), \quad (111)$$

$$\hat{\Delta}_j^-(\varepsilon) = u_j(x_j^* + \lambda_j \varepsilon, x_i^*, x_{-ij}^*) - u_j(x_j^* - \lambda_j \varepsilon, x_i^*, x_{-ij}^*). \quad (112)$$

Straightforward calculations deliver

$$\frac{\partial \hat{\Delta}_i^+(0)}{\partial \varepsilon} = \lambda_i \frac{\partial u_i(x_N^*)}{\partial x_i} = 0, \quad (113)$$

$$\frac{\partial^2 \hat{\Delta}_i^+(0)}{\partial \varepsilon^2} = \lambda_i \left(\lambda_i \frac{\partial^2 u_i(x_N^*)}{\partial x_i^2} + 2\lambda_j \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} \right), \quad (114)$$

and

$$\frac{\partial \hat{\Delta}_j^+(0)}{\partial \varepsilon} = -2\lambda_j \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \quad (115)$$

$$\frac{\partial^2 \hat{\Delta}_j^+(0)}{\partial \varepsilon^2} = -4\lambda_j \lambda_i \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j} > 0. \quad (116)$$

Similarly,

$$\frac{\partial \hat{\Delta}_i^-(0)}{\partial \varepsilon} = -\lambda_i \frac{\partial u_i(x_N^*)}{\partial x_i} = 0, \quad (117)$$

$$\frac{\partial^2 \hat{\Delta}_i^-(0)}{\partial \varepsilon^2} = \lambda_i \left(-\lambda_i \frac{\partial^2 u_i(x_N^*)}{\partial x_i^2} + 2\lambda_j \frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} \right), \quad (118)$$

and

$$\frac{\partial \hat{\Delta}_j^-(0)}{\partial \varepsilon} = 2\lambda_j \frac{\partial u_j(x_N^*)}{\partial x_j} = 0, \quad (119)$$

$$\frac{\partial^2 \hat{\Delta}_j^-(0)}{\partial \varepsilon^2} = 0, \quad (120)$$

$$\frac{\partial^3 \hat{\Delta}_j^-(0)}{\partial \varepsilon^3} = 2\lambda_j^3 \frac{\partial^3 u_j(x_N^*)}{\partial x_j^3} > 0. \quad (121)$$

Choose now $\lambda_i = 1$, and $\lambda_j > 0$ sufficiently large so that the bracket terms in (114) and (118) are positive. Then, second-order and third-order Taylor approximations at $\varepsilon = 0$ show that $\hat{\Delta}_i^+(\varepsilon) > 0$, $\hat{\Delta}_j^+(\varepsilon) > 0$, $\hat{\Delta}_i^-(\varepsilon) > 0$, and $\hat{\Delta}_j^-(\varepsilon) > 0$ all hold for $\varepsilon > 0$ small enough, which yields the desired contradiction. In the second case,

$$\frac{\partial^2 u_i(x_N^*)}{\partial x_j \partial x_i} < 0 < \frac{\partial^2 u_j(x_N^*)}{\partial x_i \partial x_j}, \quad (122)$$

and consequently $\partial^3 u_j(x_N^*)/\partial x_j^3 < 0$. It is now straightforward to check that, again for $\lambda_i = 1$ and $\lambda_j > 0$ sufficiently large, inequalities $\hat{\Delta}_i^+(\varepsilon) < 0$, $\hat{\Delta}_j^+(\varepsilon) < 0$, $\hat{\Delta}_i^-(\varepsilon) < 0$, and $\hat{\Delta}_j^-(\varepsilon) < 0$ all hold for $\varepsilon > 0$ small enough. The remainder of the argument stays as before. This completes the proof. \square

Proof of Proposition 5 It will be shown that P is a weighted potential. Since any weighted potential is, in particular, a generalized ordinal potential, this is sufficient to prove the proposition. So consider some player $i \in N$, prices $p'_i \geq 0$ and $p''_i \geq 0$, as well as a price vector $p_{-i} \in \mathbb{R}_+^{n-1}$. It is claimed that

$$P(p'_i, p_{-i}) - P(p''_i, p_{-i}) = \frac{\hat{w}_i}{1 + 2s_i c_i} \cdot (u_i(p'_i, p_{-i}) - u_i(p''_i, p_{-i})). \quad (123)$$

To see this, note first that

$$u_i(p_i, p_{-i}) = \left(p_i - c_i \left(Q_i - s_i p_i + \sum_{j \neq i} \theta_{ij} p_j \right) \right) \left(Q_i - s_i p_i + \sum_{j \neq i} \theta_{ij} p_j \right) \quad (124)$$

$$\begin{aligned}
&= (1 + 2s_i c_i) Q_i p_i - (1 + s_i c_i) s_i p_i^2 + (1 + 2s_i c_i) \sum_{j \neq i} \theta_{ij} p_i p_j \\
&\quad + \{ \text{terms constant in } p_i \}
\end{aligned} \tag{125}$$

$$\begin{aligned}
&= (1 + 2s_i c_i) p_i \left\{ Q_i \left(1 - \frac{p_i}{2p_i^0} \right) + \sum_{j \neq i} \theta_{ij} p_j \right\} \\
&\quad + \{ \text{terms constant in } p_i \}.
\end{aligned} \tag{126}$$

Therefore,

$$\begin{aligned}
u_i(p'_i, p_{-i}) - u_i(p''_i, p_{-i}) &= (1 + 2s_i c_i) Q_i \left\{ p'_i \left(1 - \frac{p'_i}{2p_i^0} \right) - p''_i \left(1 - \frac{p''_i}{2p_i^0} \right) \right\} \\
&\quad + (1 + 2s_i c_i) \sum_{j \neq i} \theta_{ij} (p'_i - p''_i) p_j.
\end{aligned} \tag{127}$$

On the other hand,

$$\begin{aligned}
P(p'_i, p_{-i}) - P(p''_i, p_{-i}) &= \hat{w}_i Q_i \left\{ p'_i \left(1 - \frac{p'_i}{2p_i^0} \right) - p''_i \left(1 - \frac{p''_i}{2p_i^0} \right) \right\} \\
&\quad + \hat{w}_i \sum_{j=1}^{i-1} \theta_{ij} (p'_i - p''_i) p_j + \sum_{j=i+1}^n \hat{w}_j \theta_{ji} p_j (p'_i - p''_i).
\end{aligned} \tag{128}$$

We will show below that

$$\hat{w}_j \theta_{ji} = \hat{w}_i \theta_{ij} \quad (j = i + 1, \dots, n). \tag{129}$$

Using this in (128), and recalling (127), we get

$$\begin{aligned}
P(p'_i, p_{-i}) - P(p''_i, p_{-i}) &= \hat{w}_i Q_i \left\{ p'_i \left(1 - \frac{p'_i}{2p_i^0} \right) - p''_i \left(1 - \frac{p''_i}{2p_i^0} \right) \right\} \\
&\quad + \hat{w}_i \sum_{j \neq i} \theta_{ij} (p'_i - p''_i) p_j
\end{aligned} \tag{130}$$

$$= \frac{\hat{w}_i}{1 + 2s_i c_i} (u_i(p'_i, p_{-i}) - u_i(p''_i, p_{-i})), \tag{131}$$

as claimed. So it remains to be checked that (129) holds true, or equivalently, that

$$\begin{aligned}
&|\theta_{12}| \cdot \dots \cdot |\theta_{j-1j}| \cdot |\theta_{j+1j}| \cdot \dots \cdot |\theta_{nn-1}| \cdot \theta_{ji} \\
&= |\theta_{12}| \cdot \dots \cdot |\theta_{i-1i}| \cdot |\theta_{i+1i}| \cdot \dots \cdot |\theta_{nn-1}| \cdot \theta_{ij} \quad (j = i + 1, \dots, n).
\end{aligned} \tag{132}$$

But this was shown in the proof of Theorem 1. Since $\hat{w}_i > 0$ for $i \in N$, we conclude that P is indeed a weighted potential for the differentiated Bertrand game. The proposition follows. \square

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